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Haar Wavelets and Multiple-Scale Pascal Polynomial Triangle Methods for Solving Nonlinear Volterra Integral Equations

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Abstract:

The aim of this paper is to propose a numerical approximation method to solve nonlinear Volterra integral equations of the second kind using Haar wavelets and multiple-scale Pascal polynomial methods. These methods are specifically derived for nonlinear problems. In this numerical approximation, we do not need to use numerical integration which is one of the advantages of our proposed method. Numerical examples are tested to demonstrate the validity of the method and the efficiency of the method is confirmed.

Key Words: Nonlinear, Volterra integral equations, Haar wavelets method, multiple-scale method, Pascal polynomial 2020 MSC: 45G10, 45D05, 65T60, 34E13, 65R20, 65D15.

الملخص:

الهدف من هذه الاطروحة هو اقتراح طريقة تقريب عددي لحل معادلات فولتيرا التكاملية اللاخطية من النوع الثاني باستخدام موجات هار وباسكال متعددة المقاييس .هذه الأساليب مشتقة خصيصا للمشاكل غير الخطية .في هذا التقريب العددي، لا نحتاج إلى استخدام التكامل العددي الذي يعد واحدًا من مزايا طريقتنا المقترحة .يتم اختبار الأمثلة العددية لإثبات صدقها تم تأكيد كفاءة الطريقة.

الكليمات المفتاحية : معادلات فولتيرا غير الخطية ، طريقة هار المويجات ، طريقة المقاييس المتعددة ، باسكال متعدد الحدود 2020 MSC: 45G10، D0545، T6065، E1334، R2065، D1565

پوختە:

ئامانجی ئەم توێژینەوەیە پێشنیارکردنی شێوازێکی نزیکبوونەوەی ژمارەیی بۆ چارەسەرکردنی هاوکێشە ناهێڵییە تەواوکارییەکانی قۆڵتێرا لە جۆری دووەم بە بەکارهێنانی ڕێگەی شەپۆلەکانی هار و پاسکالی فرە پێوەر. ئەم ڕێگایانە بە تایبەتی بۆ کێشە ناهێڵییەکان وەرگیراون. لەم نزیکبوونەوەی ژمارەییەدا پێویست ناکات تەواوکاریی ژمارەیی بەکاربهێنین کە یەکێکە لە باشییەکانی ڕێگەکەی ئێمە. نموونەی ژمارەیی تاقی دەکرێنەوە بۆ نیشاندانی ڕەوایی و ڕێگەکە و کارایی شێوازەکە چوستی ڕێگەکەش پشتڕاست دەکرێتەوە.

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1. Introduction

In recent decades Integral Equations (IEs) arose in many fields, especially in fluid mechanics, biological models, solid-state physics, kinetics in chemistry, etc. Consider the Volterra integral equation (VIE) of the second kind given by:

$$u(x) = f(x) + \int_a^x k\big(x,t,u(t)\big)dt, x \in I \coloneqq [a,b] \quad (1)$$

Where $u\colon [a,b] \to \mathbb{R}$ is the unknown function, $k\colon T \times \mathbb{R} \to \mathbb{R}$ with $T = \{(x,t)\colon a \le t \le x \le b\}$ and $f\colon [a,b] \to \mathbb{R}$ are the given functions. To obtain $u(x)$, we have supposed that the equation (1) has unique solution. It has well known, that diffusion problem, electroelastic, heat conduction, and many other areas of science give place to nonlinear IEs [1,2,3]. In most cases of equation (1), analytical solutions are either difficult to find or do not exist, so numerical methods are needed for finding the approximated solution of nonlinear IEs. Lately, many researchers had found solutions for nonlinear VolterraFredholm integral equations by different methods. In [4,5,6,7] the Taylor collocation method was intended for finding a solution for the nonlinear Volterra-Fredholm IE's in terms of Taylor polynomials. The method changed the IE into an equation of matrix by the collocation points. El-Ameen and El-Kady in [8] proposed a direct method that the Volterra-Fredholm Hammerstein reformulated by a Fredholm IE and then it is converted to an integral equation of equation (1). Aziz and Sirajul-Islam [9] have used the special characteristics of Haar wavelets in one and two dimensions and they can calculate Haar constants without even solving the obtained equation's system. Sirajul-Islam et al. in [10] proposed a novel technique established on Haar wavelets that the equation (1) expressed by a $(2M)^2$ system of nonlinear equations and then the systems have been solved by applying Newton's or Broyden's method. Mashayekhi et al.[11] solved nonlinear mixed VolterraFredholm IEs by using hybrid functions that consist of Bernoulli polynomial and block-pulse functions. Deniz in [12] has used a modification of the optimal perturbation iteration method to solve the nonlinear VIEs. Next, Linear and nonlinear IEs have been solved by Legendre multi-wavelets collocation method in [13]. Sathar et al. [14] presented a numerical technique based on a mix of Haar Wavelets Methods and Newton-Kantorovich to solve second kind nonlinear Volterra-Fredholm IEs. Almasoodi et al. [15] constructed a method from a group of general linear methods and a special quadrature rule from natural Runge-Kutta methods to solve VIEs. The two-point Taylor formula, which is a special case of the Hermite interpolation, was presented by Karamollahi et al. [16] as a numerical approximation approach. Hernández-Verón et al. [17] approximated the solution of Hammetsteintype of nonlinear IEs when the non separable property of the kernel was supposed. Most of the above authors have used two methods for solving equation (1) or its similar. In our work, we construct a novel approach to compute the solution of equation (1) numerically from Haar wavelets method approximating $u(x)$ and multiple-scale Pascal polynomial for expanding the $k(x,t,u(t))$. Multiple-scale Pascal polynomial was used for solving inverse Cauchy problems in [18,19] and Haar Wavelete is an easy and simple method for approximating the exact solutions of a problem. An advantage of our technique is do not need to apply numerical integration and it is easy in calculation.

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This paper is organized as:

2. Establishing Haar Wavelets and Polynomial Expansion

In this section we will explain how use we the Haar wavelets and polynomial Expansion.

2.1. Haar Wavelets

Wavelet family $\left(\psi_{j,i}(x)\right)_{j\in\mathbb{N},i\in\mathbb{Z}}$ is a subfamily of the Hilbert space $L^2(\mathbb{R})$ which is orthonormal, with property that all wavelet family's functions are generated via a fixed function h noun as mother wavelet out of translations and dilations. The wavelet family holds the following condition:

$$\psi_{j,i}(x) = 2^{\frac{i}{2}}h\left(2^j x - i\right).$$

The Haar wavelet family defined on the interval $[0,1)$ contains the following mappings:

$$h_1(x) = \begin{cases} 1 & \text{for } 0 \le x < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$h_i(x) = \begin{cases} 1 & \text{for } \alpha \le x < \beta \\ -1 & \text{for } \beta \le x < \gamma \\ 0 & \text{elsewhere} \end{cases}$$

where

$$\alpha = \frac{k}{m},\ \beta = \frac{k+0.5}{m},\ \gamma = \frac{k+1}{m};\ m = 2^j, j = 0,1,\dots,k = 0,1,\dots,m-1$$

Haar wavelets are characterized by two numbers: $j \in \mathbb{Z}$ refers to the wavelet's level, k is the parameter's translation. The i, m and k's relation defined as: $i = m + k + 1$. The scaling mapping is $h_1(x)$ and the mother wavelet is $h_i(x)$ for Haar wavelet's family.

We can write any function $u(x) \in L^2(\mathbb{R})$ defined on $[0,1)$ of a form of an infinite sum of Haar wavelets as follows:

$$u(x) = \sum_{i=1}^{\infty} a_i h_i(x) \qquad (2)$$

Where

$a_i \in \mathbb{R}, \forall i.$

For approximation aim, the series (2) is truncated by considering a greatest value J of $j \in \mathbb{Z}$; dilation parameter, as:

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x) \qquad (3)$$

The integer J is then called the resolution's maximum level. We define $M = 2^J$ which also is an integer.

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To calculate the unknown coefficients a_i, we have to find the collocation points $x_p = \frac{p-0.5}{2M}, p = 1,2,\dots,2M$. Now, by substituting the collocation points x_p 's into equation (3), we get a $2M \times 2M$ linear system of equations:

$$u(x_p) = \sum_{i=1}^{2M} a_i h_i(x_p),\ p = 1,2,\dots,2M \qquad (4)$$

Which its matrix form is:

$$u(x_p) = H^t a,$$

where H is an $2M \times 2M$ with $h_{ij} = h_i(x_j)$ and the coefficients, $a = [a_1, a_2, \dots, a_{2M}]^t$. while the symmetric Haar matrix $H$is with only elements $1, -1$ or 0

2.2. Polynomial Expansion

an ill-posed problem like polynomial interpolation makes the higher-order polynomials interpolation not be an easy task to numerical computation. So to over come those difficulties, Liu and Atluri [22] have offered a special length into polynomials expansion in higher orders, which made the accuracy of the numerical results better the for the applications to solve some linear problems of ill-posed ones. In this subsection we use the multiple-scale expansion technique by high-order polynomials, which proposed by [18, 19] for solving inverse Cauchy problems and elliptic equations and can over come the ill-condition problems behavior which mentioned above.

For a test solution of nonlinear integral equations (NIEs) we use expansion of polynomial which is unpretentious and simple to conclude the wanted linear algebraic equations (LAEs) to set the coefficients expansion after an appropriate collocation of points in the area. However, it is rare.

used like a main way for solving nonlinear IEs, which usually the LAEs results major difficulty is that they are of high ill-condition problems. How to reducing the number of the conditions in the linear system is the important issue in applying expansion of polynomials method to solve the nonlinear IEs.

Now, we can regard the nonlinear part $k(x,t,u(t))$ as a two variables function can be written as:

$$k(x,t,u(t)) = \sum_{i=1}^{n} \sum_{j=1}^{i} b_{ij} x^{i-j} t^{j-1} \qquad (5)$$

Where the $n_1 = \frac{n(n+1)}{2}$ coefficents b_{ij} are to be determined. Note that the polynomial's order in equation (5) is $n-1$.

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The equation (5) formulated from the elements in polynomial matrix

$$\begin{bmatrix} 1 & y & y^2 & \dots & y^{m-1} & y^m \\ x & xy & xy^2 & \dots & xy^{m-1} & xy^m \\ x^2 & x^2y & x^2y^2 & \dots & x^2y^{m-1} & x^2y^m \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ x^m & x^my & x^my^2 & \dots & x^my^{m-1} & x^my^m \end{bmatrix} \quad (6)$$

which have been used in expanding $k(x,t,u(t))$. If the entries are tied up in left-upper triangle, then the expansion is called Pascal triangle expansion.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & & x & & y & & \\ & & & x^2 & & xy & & y^2 & \\ & & x^3 & & x^2y & & xy^2 & & y^3 \\ & x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 \\ x^5 & & x^4y & & x^3y^2 & & x^2y^3 & & xy^4 & & y^5 \\ \dots & & \dots & & \dots & & \dots & & \dots & & \dots \end{array} \quad (7)$$

…

When multiply each element of equation (7) by a scalar we get equation (5); which is known multiple scale Pascal triangle.

3. Formulation of a Linear Systems

To get a good solution of the nonlinear Fredholm, nonlinear Volterra integral equations of the second kind with a small error as much as possible; equation (1), via Haar wavelet to approximate $u(x)$ and the multiple scale Pascal triangle expansion to approimate $k(x,t,u(t))$ we need:

3.1. Nonlinear Volterra integral Equations

If the Nonlinear Volterra integral equation is supposed as

$$u(x) = f(x) + \int_a^x k(x,t,u(t))dt \quad (8)$$

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then the approximation formula for equation (8) is:

$$\begin{aligned}\sum_{i=1}^{2M} a_i h_i(x) &= f(x) + \int_a^x \sum_{i=1}^{n}\sum_{j=1}^{i} b_{ij} x^{i-j} t^{j-1} dt \\ &= f(x) + \sum_{i=1}^{n}\sum_{j=1}^{i} b_{ij}\frac{x^i}{j} - \sum_{i=1}^{n}\sum_{j=1}^{i} b_{ij} x^{i-j}\frac{a^j}{j} \\ &= f(x) + \sum_{i=1}^{n}\left(c_i x^i - d_i x^{i-1}\right) \\ &= f(x) + \sum_{i=0}^{n} C_i x^i \qquad (9)\end{aligned}$$

Where

$$\begin{aligned} c_i &= \sum_{j=1}^{n}\frac{b_{ij}}{j} \\ d_1 &= \sum_{j=1}^{n} b_{jj}\frac{(a^j)}{j} \\ d_i &= \sum_{j=1}^{n-i+1} b_{(n-j+1)(n-j-i+2)}\frac{a^j}{j}\ 2 \le i \le n-1, \\ d_n &= b_{n1} \times a \end{aligned}$$

and

$$\begin{aligned} C_0 &= d_1, \\ C_i &= c_i - d_{i+1}\ i = 1,2,\dots,n-1, \\ C_n &= c_n. \end{aligned}$$

We Select $2M$ collocation points on the interval $I = [a, b]$, we obtain a system of $2M$ equations with $2M + n + 1$ coefficients. It is suitable to write the system of equations in form of matrix- vector product as:

$$AC = f \qquad (10)$$

Where $\mathbf{A} = [H \quad -L]_{2M\times 2M+n+1}, \mathbf{C} = [a_1, a_2, \dots, a_{2M}, C_0, C_1, \dots, C_n]^t,$ and $\mathbf{f} = [f(x_1), f(x_2), \dots, f(x_{2M})]$. To obtain the values of coefficients we can write equation (10) as:

$$\mathrm{DC} = \mathrm{F} \qquad (11)$$

Where $\mathbf{D} = \mathbf{A^t A}$ and $\mathbf{F} = \mathbf{A^t f}$. Since linear system equations (11) has a normal matrix $\mathbf{D}$ of $2M + n + 1$ equations and we can use conjugate gradiant method's algorithm (CGM) to find C.

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Remark 1. There are some paper [9,14,20] have used the concept of Haar wavelete for the nonlinear part and then have used the Newton method's for two variables to find all coefficients given. And they did it without solving the system of equations which computing them needs so much time for large values of M. But for finiding each cofficient they need Newton method formula to calculate the coefficients. Some of them as in [20] did not involved numerical integration but [9,14] have approximated the integral with the Haar wavelet basis and performed exact integration of Haar functions.

4.Numerical Examples

In this section we apply our method to nonlinear Fredholm and nonlinear Volterra integral equations. Matlab (R2019a) was used (in a computer which has a RAM of 2 GB and CPU of 2GHz) to calculate each of these examples.

Example 1. We have the following equation [14, 21]:

$$u(x) = -x^2 + \frac{x}{3}(2\sqrt{2} - 1) + 2 + \int_0^1 xt\sqrt{u(t)}dt. \qquad (12)$$

The exact solution is $u(x) = 2 - x^2$.

The computer simulation gave the following table

Table 1: Maximum absolute error for nonlinear Fredholm integral equation (12)

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$M = 2^1$	0.343885790515963	0.349821914603896	0.350564219639614	0.349765931194146
$M = 2^2$	0.156791030216482	0.249828480856119	0.251227527246668	0.252200402730466
$M = 2^3$	0.083262680959299	0.124938964843750	0.125019765253166	0.125253029100295
$M = 2^4$	0.047376876668156	0.062492370605469	0.062492370605469	0.062497161939197
$M = 2^5$	0.025951710465165	0.031249046325684	0.031249046325684	0.031249046325684
$M = 2^6$	0.013723266175552	0.015624880790710	0.015624880790710	0.015624880790710
$M = 2^7$	0.007080376623809	0.007812485098839	0.007812485098839	0.007812485098839
$M = 2^8$	0.003599695214012	0.003906248137355	0.003906248137355	0.003906248137355
$M = 2^9$	0.001815389762843	0.001953124767169	0.001953124767169	0.001953124767169
$M = 2^{10}$	0.000911664432146	0.000976562470896	0.000976562470896	0.000976562470896
$M = 2^{11}$	0.000456828960097	0.000488281246362	0.000488281246362	0.000488281246362
$M = 2^{12}$	0.000277671294557	0.000244140624545	0.000244140624545	0.000244140624545

Form table (1), it is evident that the solution's maximum absolute error (norm of 'inf' have been used) decreases significantly when the collocation points number increases. The maximum absolute error of exhibits oscillating behavior when the values of n are increased.

Example 2. Let's solve following equation [9, 21]:

$$u(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)(u(t))^3 dt \qquad (13)$$

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The exact solution is $u(x) = \sin(\pi x) + \frac{20-\sqrt{391}}{3}\cos(\pi x)$.

The computer simulation gives the following table:

Table 2: Maximum absolute error for nonlinear Fredholm integral equation (13)

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$M = 2^1$	0.027135262687034	0.027105152065990	0.027067302526345	0.027043556249940
$M = 2^2$	0.014957331218283	0.033633463530456	0.033633463530456	0.033633463530456
$M = 2^3$	0.011382230925490	0.010897184532724	0.010893028169126	0.010890308195593
$M = 2^4$	0.011534360997371	0.004946910438464	0.005987365417992	0.006586350715125
$M = 2^5$	0.009181688396928	0.004136424687560	0.004793507426296	0.005107002498647
$M = 2^6$	0.006213380871455	0.003138895590868	0.003411531167843	0.003465215204366
$M = 2^7$	0.003754604372783	0.002056171998369	0.002082605670479	0.001990629586698
$M = 2^8$	0.002105758555405	0.001183835837244	0.001102518902069	0.000966165949870
$M = 2^9$	0.001126145057476	0.000613806804031	0.000507730368230	0.000501444391521
$M = 2^{10}$	0.000584717623165	0.000288759531196	0.000270543142096	0.000297722613972
$M = 2^{11}$	0.000298347016838	0.000136446404560	0.000159600603741	0.000175159095136

The maximum absolute error of the solution of equation (13) in the table (2) decreases significantly when the collocation points number increases. When the value of n is raised, the solution's error displays fluctuating behavior.

Example 3. For a nonlinear Volterra integral equation [9, 21]:

$$u(x) = \frac{3}{2} - \frac{1}{2}\exp(-2x) - \int_0^x (u(t)^2 + u(t))dt \qquad (14)$$

The exact solution is $u(x) = \exp(-x)$.

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The computer simulation of the present method gives the following table:

Table 3: The error obtained for nonlinear Volterra integral equation (14) in the present method

	$n=2$	$n=3$	$n=4$	$n=5$
$M=2^1$	0.108509877465090	0.103709123542693	0.101362910779164	0.100105593017332
$M=2^2$	0.117426632851684	0.117426632851684	0.117426632851684	0.117426632851684
$M=2^3$	0.060577077154772	0.060577077154772	0.060577077154772	0.060577077154772
$M=2^4$	0.030765513656419	0.030765513656419	0.030765513656419	0.030765513656419
$M=2^5$	0.015503405285316	0.015503405285316	0.015503405285316	0.015503405285316
$M=2^6$	0.007782041948985	0.007782041948985	0.007782041948985	0.007782041948985
$M=2^7$	0.003898628051201	0.003898628051201	0.003898628051201	0.003898628051201
$M=2^8$	0.001951218582387	0.001951218582387	0.001951218582387	0.001951218582387
$M=2^9$	0.000976085779238	0.000976085779238	0.000976085779238	0.000976085779238
$M=2^{10}$	0.000488162055261	0.000488162055261	0.000488162055261	0.000488162055261
$M=2^{11}$	0.000244110824497	0.000244110824497	0.000244110824497	0.000244110824497

We can observe that as collocation points number rises, the solution's greatest absolute error of equation (14) in the table (3) goes way down. When the value of n is increased, the error of the solution reflects the constant behavior.

Example 4. Consider the nonlinear Volterra integral equation [9]:

$$u(x) = f(x) + \int_0^x xt^2(u(t))^2 dt \qquad (15)$$

Where

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$$f(x) = \left(1 + \frac{11}{9} + \frac{2}{3} - \frac{1}{3}x^3 + \frac{2}{9}x^4\right)\ln(x+1) + \frac{1}{3}(x + x^4)(\ln(x+1))^2 - \frac{11}{9}x^2 + \frac{5}{18}x^3 - \frac{2}{27}x^4.$$

The exact solution is $u(x) = \ln(1 + x)$.

The computer simulation error of equation (15) gives the following table:

Table 4: The error obtained for equation (15).

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$M = 2^1$	0.109032170555515	0.115683087135479	0.119827503992651	0.122524774308628
$M = 2^2$	0.066901594948058	0.073074597390479	0.076648978686386	0.078860174648933
$M = 2^3$	0.041423359274478	0.044277602752303	0.045761560806697	0.046515477758269
$M = 2^4$	0.023319023180795	0.024312773007412	0.024624559002807	0.024606129036641
$M = 2^5$	0.012314390600286	0.012492783217156	0.012377261688532	0.012163921692160
$M = 2^6$	0.006244539841593	0.006161986982323	0.005999194390697	0.005837142853358
$M = 2^7$	0.003085824086050	0.002981586084752	0.002880790935835	0.002799006888463
$M = 2^8$	0.001504248668365	0.001440926360686	0.001393846496799	0.001360162908626
$M = 2^9$	0.000731815165837	0.000702262902534	0.000682952501778	0.000670971938540
$M = 2^{10}$	0.000358077171932	0.000345611337144	0.000338271586585	0.000334725478427
$M = 2^{11}$	0.000176535827489	0.000171343428942	0.000168775883700	0.000167979544551

We can observe that as collocation points number rises, so does the solution's maximum absolute error of in the table (4). When the value of n is increased for $M = 2^i, i = 1,2,3,4,5$, the solution's error increases, but when $i = 6,7, \dots ,11$, the solution's error decreases.

5. Conclusion

Our method's major benefits are its simplicity. The approach is extremely useful to solve nonlinear integral equations. In all examples in section (4), by increasing collocation points number the solution's error decreases, the suggested technique has an advantage over the majority of the references cited in this study in that it does not require numerical integration and does not include any additional algorithms to solve the nonlinear form in the equation (1). according to the runtime error our method is better than the runtime in [9,14,20,21] when the value $M = 2^i, i = 7,8, \dots ,11$, these approaches will be quite slow to achieve the approximation solution.

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